Strange attractor in optical logic cells

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ABSTRACT

Optical logic cells, employed in several tasks as optical computing or optically controlled switches for photonic switching, offer a very particular behavior when the working conditions are slightly modified. One of the more striking changes occurs when some delayed feedback is applied between one of the possible output gates and a control inpu t. Some of these new phenomena have been studied by us and reported in previous papers. A chaotic behavior is one of the more characteristic results and its possible applications range from communications to cryptography.

But the main problem related with this behavior is the binary character of the resulting signal. Most of the nowadays-employed techniques to analyze chaotic signals concern to analogue signals where algebraic equations are possible to obtain. There are no specific tools to study digital chaotic signals. Some methods have been proposed. One of the more used is equivalent to the phase diagram in analogue chaos. The binary signal is converted to hexadecimal and then analyzed.

We represented the fractal characteristics of the signal. It has th e characteristics of a strange attractor and gives more information than the obtained from previous methods. A phase diagram, as the one obtained by previous techniques, may fully cover its surface with the trajectories and almost no information may be obtained from it. Now, this new method offers the evolution around just a certain area being this lines the strange atractor.

Keywords: fractal representation, chaotic behaviour, digital chaos

1. INTRODUCTION

The analysis of chaotic behavior has been the obj ect of considerable interest in the last years. Most of the published papers deal with chaos obtained either from a numerical method or from some type of circuit, electronic or photonic. The obtained and analyzed chaos has, in both cases, analog characteristics. Numerical results, obtained from computer studies, are always related with analytical equations and conventional mathematical methods. In this way, a large collection of measure determinations have bee proposed and employed. In the same way, experimental chaotic situations are related with the behavior of analog circuits, both electronic and photonic, offering analog signals. Examples are well known, ranging from the classical Chua's circuit[1] to chaotic situations obtained from optical bistable circ uits or semiconductor lasers.

Situation is very different when a certain type of digital chaos is obtained from a digital circuit. The first difference is that, in some occasions, there is no simple equation able to determine the behavior of the circuit. Possible equations are no analytical and, hence, conventional methods employed in previously mentioned situations are no longer valid. The second difference is that employed measure methods are, in some occasions, impossible to be used. The phase diagram gives an example. If we try to represent the evolution of a train of "1" and "0" bits in a phase diagram the result should be just fou points. No information should be obtained from it. Moreover, measure methods, as the Lyapunov exponent, are ver difficult to be employed. It is possible to say, from this point of view, that some other techniques are needed when optical logical circuits show nonlinear evolution in time. Some of these techniques have been reported and presented by us in several papers[2-7]. In this occasion we are going to offer a new type of behavior in optical logic cells showing the presence of a strange atractor. The paper will present the origin of the nonlinear behavior, the method to be employed to represent this type of phenomena and the obtained results.

2. OPTICAL LOGIC CELL

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Although we have reported this cell in several places, in order to place the next results in a correct context, some words will be written about its main characteristics.

The basis is the use of an Opticall -Programmable Digital Circuit, reported previously by us[2, 7], and able to process two input binary signals. Its two outputs are logical functions of these inputs. The type of processing is related to the eight main Boolean Functions, namely, AND, OR, XOR, NAND, NOR, XNOR, ON and OFF. The programmable ability of the two outputs, as it has been described, allows the generation of several data coding for optical transmission. Moreover, as it was shown, this circuit has the possibility to generate periodic and chaotic solutions. A precise analysis of the output characteristic versus the main variable parameters, as control signal level and data signal level, has been reported[8]. With this configuration, the above-mentioned digital character of the signal is directly obtained. Two devices, P and Q with non-linea behaviors, compose it. The outputs of each one of them correspond to the two final outputs, O_1 and O_2, of the cell. Four are the possible inputs to the circuit. Two of them are for input data, $\mathbf{I}_1$ e $\mathbf{I}_2$, and the other two, ***g*** and ***h***, for control signals. The way these four inputs are arranged inside the circuit may be seen in [2-4]. A practical implementation we have carried out of the processing element has been based on an optoelectronical configuration. Optical inputs arriving to the individual devices are multilevel signals. Device Q, corresponds to a thresholding or switching device, and device P is a multistate device. The response of P is similar to the behavior of a SEED device.

A non-linear behavior can be expected from the above reported cell, if some kind of feedback should be applied. The feedback we have applied to the system, among the different possibilities, is the one corresponding from the output O_2 of Q-device to the control input, *g*, of P-device. No other additional control signal has been used. A chaotic output is obtained when the internal response time equal to zero or much smaller than the external one. We have reported some results in previously mentioned papers.

3. PHASE DIAGRAM REPRESENTATION OF DIGITAL CHAOTIC SIGNALS

As it has been pointed out above, one of the principal obstacles to represent the signals obtained from logic cells is thei binary form. The way adopted by us was to convert these signals from binary to hexadecimal form. Any train of bits is divided into groups of four bits and its corresponding hexadecimal number designs each group. The sequence is now a multilevel signal with sixteen possible values. This new situation gives as the possibility to represent the train of pulses in an almost conventional phase diagram. We represent a value of the sequence, at the ordinate axis, as a function of the previous value in the sequence, that is represented at abscises axis. In this way, any digital signal may be represented by its corresponding pattern in this diagram. If the signal has a periodic character, the trajectory will be a closed one. If the signal is a random signal, the trajectory will cover every point at the phase space representation. If the signal is a chaotic one, the resulting line will offer a pattern that will depend on the chaos characteristics. Figs. 1 and 2 offer two phase diagrams. Fig. 1 corresponds to a periodic train of bits and Fig. 2 to a chaotic one.

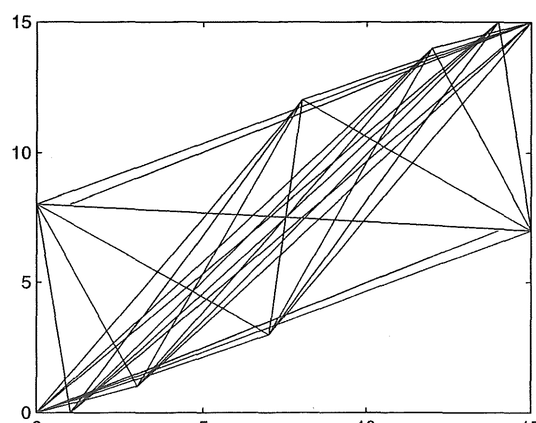

Fig. 1.- Phase Diagram of a periodic signal

These diagrams are valid to indicate the general behavior o f the system. To differentiate a periodic train of pulses from an irregular sequence of bits is useful as indication of the possible regularity of the signal. Moreover, it gives an indication about how the different groups of four digits are distributed in the phase diagram. A zone with a large

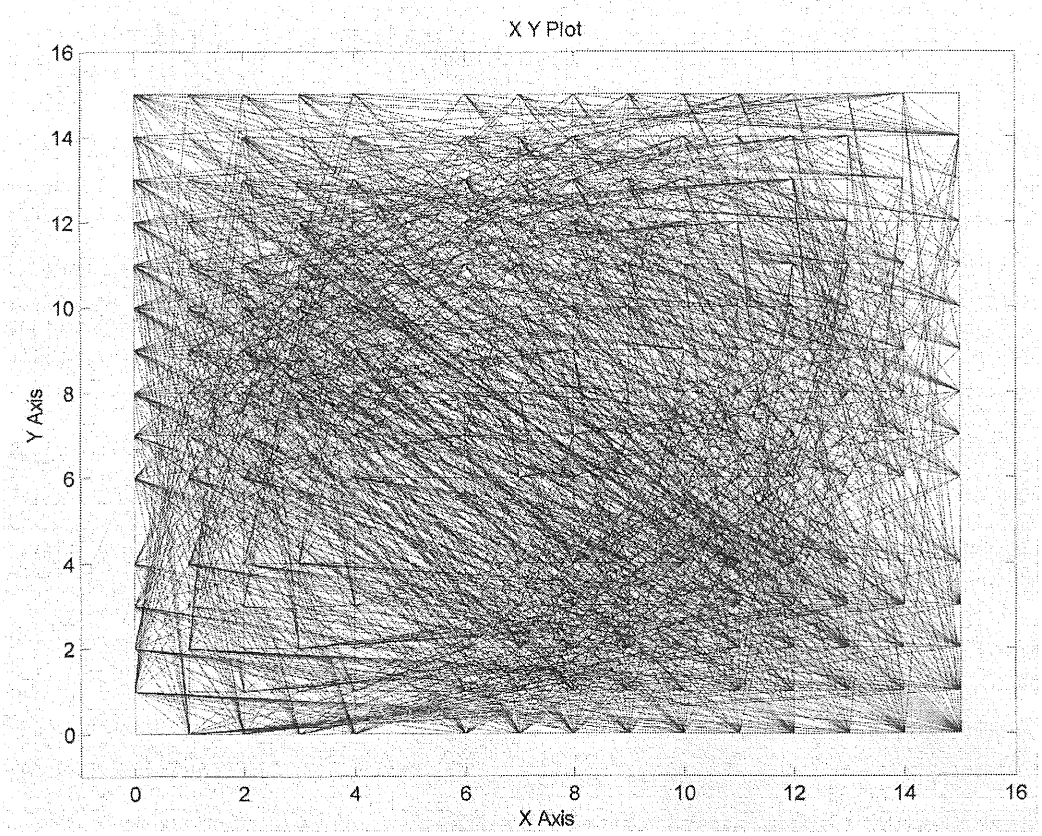

Figure 2.- Phase Diagram of Digital Chaos Signal.

concentration of occupied points gives the possibility to know the more frequently employed region and, as a consequence, which groups of bits have a larger probability to appear. But more information should be kno n in some occasions. And, hence, it is important to perform some other type of representation. This will be the fractal representation that will be presented in the next section.

4. FRACTAL REPRESENTATION OF CHAOTIC SIGNALS

4.1 General Basis of Fractal represe ntation: Iterated Maps

To get an idea of this technique we start from the properties of linear maps. These maps take the form

$$A' = TA$$

Where, for 2-D planar maps, A represents a point in the initial area and A' represents the new point under the matrix operation T:

$$T = \begin{Vmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{Vmatrix}$$

A linear map can contract or expand, rotate, shear, or reflect and area collection of points. Under suitable assumptions, repeated applications of the mapping $A_{n+1} = WA_n$ lead to an attractor. This means that starting f om different sets A_1, A'_1 one ends up with the same set A. After many iterations, each point in the initial set A_1 undergoes an orbit.

The following map gives another example of the fractal set generation in the plane:

$$w_{00}(x,y) = ((1/2)x, (1/2)y)$$
$$w_{01}(x,y) = ((1/2)x, (1/2)y + 1/2)$$
$$w_{10}(x,y) = ((1/2)x + 1/2, (1/2)y)$$
$$w_{11}(x,y) = ((1/2)x + 1/2, (1/2)y + 1/2)$$

Transformations are, in this case, labeled in a binary code, i.e., 00, 01, 10, and 11 instead 0, 1, 2, and 3. We will employ this type of transformation in this paper.

This method of generating a fractal can be computationally very time consuming, because at every iteration cycle all linear maps must act on all the points which define A. Another method, however, makes use of the chaotic nature of the orbits in this iteration process and generates the corresponding points in a more direct way.

4.2 Fractal Representation of Chaotic Signals

According to the general ideas of previous section, a new type of representation for digital chaotic signals can be introduced. In this case, a square with side of value 1 is divided into four squares with a surface 1/4 of the initial one. Each one of these new squares has a transformation assigned. The transformations are

a. Transformation {00} corresponds to

$$X' = 1/2\, X$$
$$Y' = 1/2\, Y$$

b. Transformation {01} corresponds to

$$X' = 1/2\, X$$
$$Y' = 1/2\, Y + 1/2$$

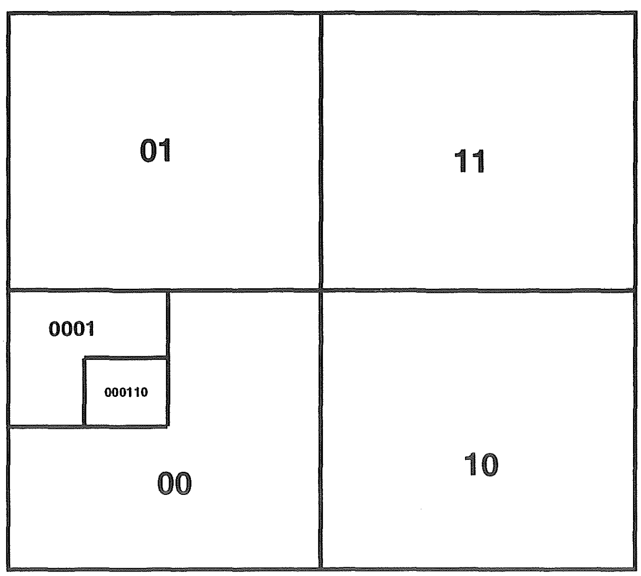

Fig. 3.- Addresses for the fractal representation

c. Transformation {10} corresponds to

$$X' = 1/2\,X + 1/2$$
$$Y' = 1/2\,Y$$

d. Transformation {11} corresponds to

$$X' = 1/2\,X + 1/2$$

$$Y' = 1/2\,Y + 1/2$$

This representation appears in Fig. 3.

The first step to construct the fractal representation is grouping the groups of eighth data in sets of two bits. Fo instance

00011011 01101000 → 00 01 10 11 01 10 10 00

These data have to be inserted now in the initial square. The initial point is the point (0, 0) although any point ma be equivalent. This implies that we are approaching 0001 1011 to 0001101100. This implies that groups of bits are represented by points and no as surfaces as well as it is not possible that the square had two different transformations.

Transformations are applied in sense inverse. That means the initial group to apply is the last group of two bits.

If these concepts are applied to our present example, we obtain

Transformation 11

$$X1 = 1/2 * 0 + 1/2 = 1/2 \quad ; Y1 = 1/2 * 0 + 1/2 = 1/2 \qquad (0.5, 0.5)$$

Transformation 10

$$X2 = 1/2 * 0.5 + 1/2 = 0.75 \quad ; Y2 = 1/2 * 0.5 = 0.25 \qquad (0.75, 0.25)$$

Transformation 01

$$X3 = 1/2 * 0.75 = 0.375 \quad ; Y3 = 1/2 * 0.25 + 1/2 = 0.625 \qquad (0.375, 0.625)$$

Transformation 00

$$X4 = 1/2 * 0.375 = 0.1875 \quad ; Y4 = 1/2 * 0.625 = 0.3125 \qquad (0.1875, 0.3125)$$

The form to apply this type of transformation will be given, in our present situation, by the string of bits “0” and “1” obtained from the chaotic structure.

5. APPLICATION TO THE SIGNAL OBTAINED FROM AN OPTICALLY PROGRAMMABLE LOGIC CELL

The above concepts may be applied to the chaotic situation pointed out previously. As it was stated, the signal of an Optically Programmable Logic Cell is periodic or chaotic according to the characteristics of the internal and external delays as well to the relation between them. To try to obtain data from a string of binary information units requires a different type of concepts than the employed in conventional analog signals. It was because that we employed in previous papers the method to convert the binary signals in hexadecimal. The method was to group the bits en sets of four bits and to convert them into their correspondent hexadecimal value. These groups of four bits were taken in consecutive order. That means that two consecutively obtained hexadecimal numbers corresponded to two sets of four bits in the same consecutive order in time. No overlapping of bits was taken.

This grouping of four bits has the advantage of working with signals easier to handle than conventional "1" and "0" bits. But if our objective is to get a situation as much close to the analog one as possible, grouping in sets of eight digits should be better. Moreover, there is an additional advantage. If one wishes to study a long train of bits, to divide this number

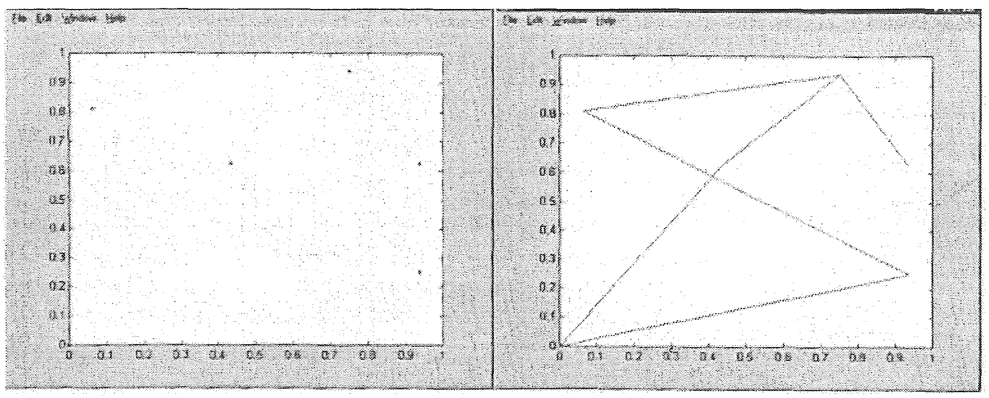

Fig. 4.- Fractal representation of a sequence from the OPLC with external delay time of 10 time units and without overlapping. Groups of 8 bits have been taken.

by four implies a reduction in computer time. The division in groups of eigh t bits should reduce again the time. Hence, in this occasion we are going to adopt a new representation of the chaotic sequence converting the binary signal to an eight-level signal. Numbers are now in a basis of order 8.

There is another possibility to handle the sequence to be studied. The analysis of the characteristics of an sequence of bits, in the sense pointed out before, may give some additional information, in some occasions, if the grouping of four or eight bits is performed in a different way. Instead to get sequences of four or eight consecutive bits, it should be possible to group them in such a way that, for example, a bit is common in forming two consecutive hexadecimal numbers. The last bit of a certain four or eight digits group being the first one of the next group. This operation will call "one bit overlapping". The overall characteristics of a chaotic signal should be independent on the type of construction we have adopted for the hexadecimal number. But it is possible that the analysis of the time evolution could change from a configuration to another. Because we have not studied this fact in previous papers, it will be the object of a particula attention in the present one.

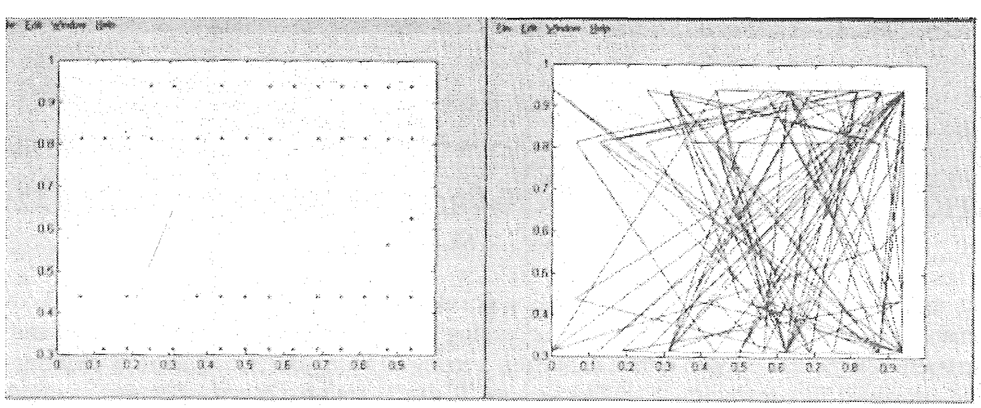

Fig. 5.- Fractal representation of a sequence from the OPLC with external delay time of 12,000 time units and without overlapping. Groups of 8 bits have been taken.

We employed the "Phase diagram" representation, indicated in previous paragraphs, as the conventional representation in previous papers. We are going to compare it now with the new fractal representation studied previously.

With these ideas in mind we have analyzed a sequence of 10^6 bits obtained from two initial conditions in the Optically Programmable Logic Cell. The first one was with an external delay time of 10 time units and the second one with a delay of 12,000 time units. The first studied case was just to indicate the importance of the relation between the external and the internal delay times. In both cases, the internal time was fixed to a very small value.

In the first case, of an external delay time of 10, if groups are of eight bits, as it was pointed out before, the results are shown in Fig. 4 (a)-(b). Fig. 4 (a) shows the obtained points and Fig. 4 (b) the trajectory follows by these points.

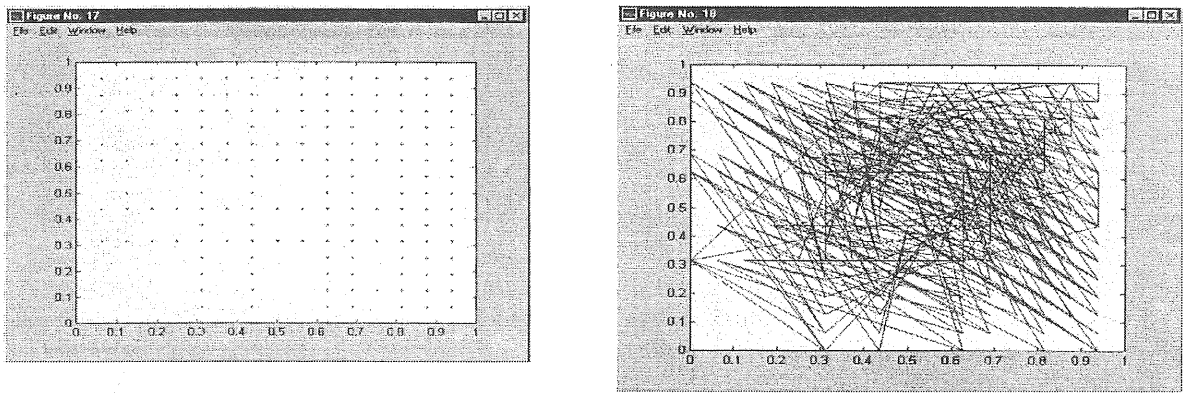

Fig. 6.- Fractal representation of a sequence from the OPLC with external delay time of 12,000 time units and with one bit overlapping. Groups of 8 bits have been taken.

The situation is very different if we change the external delay time to a longer time interval, in this case, 12345 time units. This number gives a much large relation between the external and the internal delay times. This situation corresponds to a condition where the chaotic solution has more possibilities to appear or, at least, the periodic solution has a longer period. The obtained results appear in Fig. 5 (a) and (b).

It is possible to see that the number of working points has increased with respect to the previous situation. The diagram corresponding to trajectories of consecutive shows a very irregular configuration but no indication about any fractal situation. This is not an indication that there is not a chaotic situation but it is not possible either to indicate the existence of chaos.

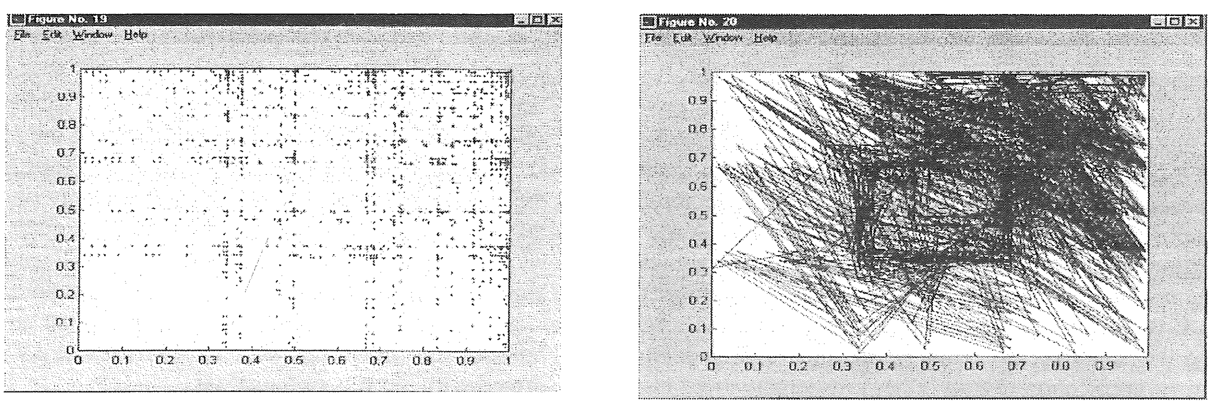

Fig. 7.- Fractal representation of a sequence from the OPLC with external delay time of 12,000 time units and with one bit overlapping. Groups of 16 bits have been taken.

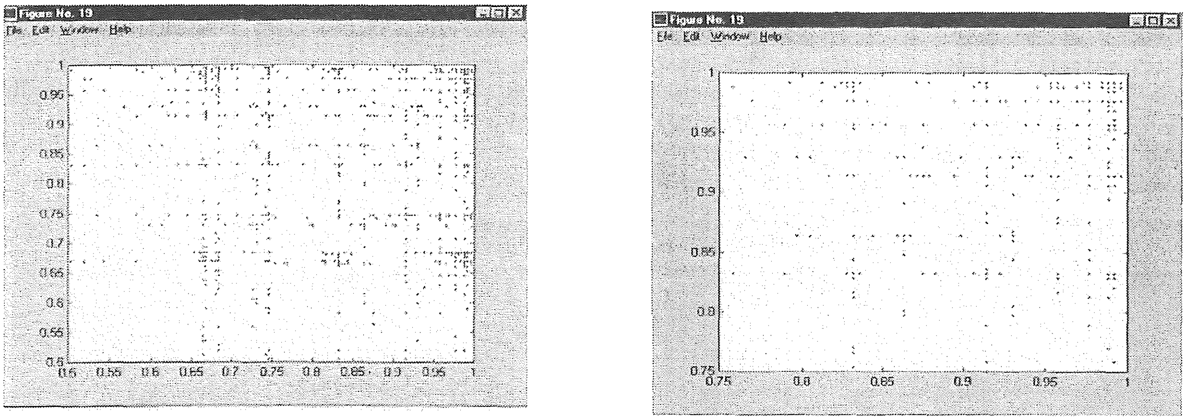

Fig. 8.- Fractal representation of a sequence from the OPLC with external delay time of 12,000 time units and with one bit overlapping. Groups of 16 bits have been taken. Fig. 8 (a) corresponds to the upper right corner of Fig. 7 (a) and Fig. 8 (b) to the upper corner of Fig. 8 (a).

If the same situation is adopted now, but with a single 1-bit overlapping, results are clearly different. They are shown in Fig. 6 (a)-(b).

The first difference with respect to the previous results is that the number of points is much higher than before. This indicates a higher complexity but nothing else. But if we look to the trajectories representation a certain indication of a region where the lines are closer together appears. Moreover there is a certain image similar to equivalent representation fo analog signals. This may be an indication of the presence of a possible fractal structure.

To analyze more carefully this result, we have extended the number of bits to 16. In this way, the fractal representation adopts a larger number of related points. The results are shown in Fig. 7 (a) – (b).

Situation now is drastically different. In the first place, the number of points has increased considerably. If one compares the result of Fig. 7 (a) with the obtained in Fig. 5 (a) we see that the points now are covering large regions of the plane. Moreover, the trajectory covers almost a whole region. This configures clearl a certain "digital" fractal.

The way to prove that this is a real fractal should be to analyze more closely a certain region of the obtained figure and to see if the same configuration repeats the structure that appears in the initial figures. We have done this and the results are shown in Figs. 8 (a) – 8 (b). We have expanded the region corresponding to the upper right corner in Fig. 8 (a) and similar zone in Fig. 8 (b). It may be appreciated that the internal structure is the same in the three figures.

6. CONCLUSIONS

The method reported in this paper represents a new way to analyze the properties of any digital chaotic signal. The previously reported methods were able to analyze the properties of analog signals but they were very difficult to apply to binary signals. We have analyzed this problem, in previous papers, by the use of a phase diagram representing the system state at a certain time as a function of the previous instant of time. This representation gave an idea about the chaotic characteristics of the signal. In some cases, the trajectory covered the whole surface of the diagram and the only obtained information was that the signal got any possible value. The fractal representation introduced in this work gives more information about the real properties of the chaotic signal and allows differentiating between a cyclic or periodic signal a chaotic one. In this case, obtained trajectories cover just a part of the diagram. If one closely observes the properties of a region, it is possible to see that a certain structure is repeated again and again with fractal characteristics. If we add this result to the fact that the signal never repeats, we conclude that the obtained image has the characteristics of a strange atractor. Its application to some other similar situations may be performed with the reported method.

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